

GROUP EXTENSIONS AND COHOMOLOGY FOR LOCALLY COMPACT GROUPS. IV

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ABSTRACT. In this paper we shall apply the cohomology groups constructed in [14] to a variety of problems in analysis. We show that cohomology classes admit direct integral decompositions, and we obtain as a special case a new proof of the existence of direct integral decompositions of unitary representations. This also leads to a Frobenius reciprocity theorem for induced modules, and we obtain splitting theorems for direct integrals of tori analogous to known results for direct sums. We also obtain implementation theorems for groups of automorphisms of von Neumann algebras. We show that the splitting group topology on the two-dimensional cohomology groups agrees with other naturally defined topologies and we find conditions under which this topology is T_2 . Finally we resolve several questions left open concerning splitting groups in a previous paper [13].

1. This paper will rely heavily on the development in [14] of cohomology groups whose properties were defined and discussed there. The object of this paper is to apply these constructions to a variety of problems in analysis. We obtain for instance a general theorem which guarantees the existence of direct integral decompositions of cohomology classes in appropriate contexts. This will contain for instance as a special case the existence of direct integral decompositions of unitary representations. This will also give a kind of Frobenius reciprocity theorem for our induced modules, and in addition we obtain a splitting theorem for direct integrals of tori, analogous to the known theorems for products of tori. Also we use this to study groups of inner automorphisms of von Neumann algebras, and obtain an implementation theorem stated entirely in terms of von Neumann algebras.

We also will study the topology on $\underline{H}^2(G, A)$ and relate it to the splitting group topology defined in [11], and investigate some sufficient conditions for groups $\underline{H}^q(G, A)$ to be Hausdorff or in other words for the boundaries to be closed. We also investigate briefly some aspects of H^1 for amenable groups acting linearly, and comment on a counterexample against further extensions, and the

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existence of invariant measures. Finally we shall take up several questions concerning universal covering groups as defined in [13], and prove several theorems stated in [13] with the comment that their proofs would be given in a subsequent paper.

2. Suppose that G is locally compact separable, $A \in \mathcal{P}(G)$ and X a measure space as in [14]. We form the group $U(X, A)$ and let G act as usual by $(g \cdot F)(x) = g \cdot (F(x))$ so that $U(X, A) \in \mathcal{P}(G)$. Since this module is a direct integral of copies A , and since cohomology commutes with direct sums, one expects an analogous result that cohomology commutes with direct integrals. This is the case, but there is one minor point to be clarified and that is the occurrence of such groups as $U(X, \underline{H}^n(G, A))$. This is well defined if $\underline{H}^n(G, A)$ is Hausdorff, but if it is not we view it as a pseudo-polonais group given by the triple $(\underline{C}^{n-1}(G, A), \underline{Z}^n(G, A), \underline{\delta}^n)$ as in [14, §2]. There $U(X, \underline{H}^n(G, A))$ was defined as the cokernel of the induced map $U(X, \underline{C}^{n-1}(G, A)) \rightarrow U(X, \underline{Z}^n(G, A))$ with the quotient topology. Recall that it was shown in [14, Proposition 10] that the closure of the identity in a topological group such as $U(X, \underline{H}^n(G, A))$ consists of all functions from X in $\underline{H}^n(G, A)$ taking values a.e. in the closure of the identity element of $\underline{H}^n(G, A)$. We have now the following fact.

THEOREM 1. *There are isomorphisms of topological groups $\underline{H}^n(G, U(X, A)) \simeq U(X, \underline{H}^n(G, A))$.*

PROOF. The cohomology groups on the left are defined as those of a co-chain complex $\underline{C}^*(G, U(X, A))$, and the n -dimensional group of this complex is exactly $U(G^n, U(X, A))$ which can be identified to $U(G^n \times X, A)$ by our Fubini theorem. On the other hand, the groups on the right-hand side are defined as certain cokernels of maps as defined above, but a moment's reflection shows that they are equivalently described as cohomology groups of a complex $U(X, \underline{C}^*(G, A))$ where the n -dimensional group is $U(X, \underline{C}^n(G, A))$ with coboundary operators Δ^n defined pointwise by $(\Delta^n F)(x) = (\delta^n F)(x)$. Again, by our Fubini theorem $U(X, \underline{C}^n(G, A)) = U(X, U(G^n, A))$ is isomorphic as a topological group to $U(X \times G^n, A)$. Moreover, one checks that under this identification Δ^n goes over into the coboundary coming from the complex $\underline{C}^*(G, U(X, A))$. Thus the two groups in the statement of the theorem are the cohomology groups with the correct topologies defined by two complexes of polonais groups which are algebraically and topologically isomorphic. The desired result follows at once.

We have several immediate consequences.

COROLLARY 1. *If $\underline{H}^n(G, A)$ is Hausdorff then $\underline{H}^n(G, U(X, A))$ is Hausdorff and isomorphic to $U(X, \underline{H}^n(G, A))$ defined in the usual way.*

For the following let us denote by $\widetilde{H}^n(G, B)$ the group obtained from

$\underline{H}^n(G, B)$ by dividing by the closure of the identity, or equivalently $\tilde{H}^n(G, B)$ is $\underline{Z}^n(G, B)$ modulo the closure of the boundaries.

COROLLARY 2. *We have isomorphisms $\tilde{H}^n(G, U(X, A)) \simeq U(X, \tilde{H}^n(G, A))$.*

PROOF. This follows by dividing each side of the statement of the theorem by the respective closure of the identity and using Proposition 10 of [14].

If we specialize A in the theorem to T , the circle group with trivial action, and take $n = 1$, we obtain an isomorphism $\text{Hom}(G, U(X, T)) \simeq U(X, \text{Hom}(G, T))$, which one recognizes as a special case of the existence of direct integral decompositions, for let $H = L^2(X)$ and note that $U(X, T)$, viewed as multiplication operators, are precisely the unitary operators commuting with the von Neumann algebra A of all multiplication operators. A unitary representation π commuting with A is precisely a homomorphism of G into $U(X, T)$ and the isomorphism above yields on the right-hand side the explicit direct integral decomposition of π , in this case into one-dimensional representations.

Direct integral theory is more complicated than this but essentially the most general situation is that of a Hilbert space $L_2(X, H_0)$ of vector valued functions in a Hilbert space H_0 with A as usual the abelian algebra of multiplication operators by scalars. Then the unitary group of all operators commuting with A is clearly realized as $U(X, V)$ where $V = U(H_0)$ is the unitary group of H_0 . A representation π commuting with A is simply a homomorphism of G into $U(X, V)$ or an element of $\underline{Z}^1(G, U(X, V))$ and its direct integral decomposition is precisely an element of $U(X, \underline{Z}^1(G, V))$. Passage to cohomology amounts to taking a unitary equivalence class of a representation $\lambda \in \underline{Z}^1(G, V)$ and to taking a restricted kind of equivalence class for $\pi \in \underline{Z}^1(G, U(X, V))$ where we conjugate only by unitaries in $U(X, V)$, which is an equivalence relation appropriate for direct integral theory. Thus to obtain direct integral theory all we have to do in a sense is replace A in Theorem 1 by a noncommutative polonais G -module and take $n = 1$, and look at cocycle and cohomology sets. The proof of such a theorem is exactly the same as the proof of Theorem 1.

THEOREM 2. *If V is any polonais G -module, commutative or not, then we have a topological isomorphism of cocycle and cohomology sets*

$$\underline{Z}^1(G, U(X, V)) \simeq U(X, \underline{Z}^1(G, V)) \quad \text{and} \quad \underline{H}^1(G, U(X, V)) \simeq U(X, \underline{H}^1(G, V)).$$

We may make use of Theorems 1 and 2 to give a version of the Frobenius reciprocity theorem for our induced modules $I_H^G(A)$ as defined in [14]. We may state and prove this for noncommutative modules as well.

THEOREM 3. *Let H be a closed subgroup of G and let U be any polonais*

H-module and let V be any locally compact G -module. Then we have an isomorphism $\text{Hom}_H(V, U) \simeq \text{Hom}_G(V, I_H^G(U))$.

PROOF. If $\phi \in \text{Hom}_H(V, U)$, we define $K(\phi) \in \text{Hom}_G(V, I_H^G(U))$ as follows: $(K(\phi)(v))(s) = \phi(s^{-1} \cdot v)$ where $s^{-1} \cdot v$ denotes the action of G on V . Then $\phi((sh)^{-1} \cdot v) = \phi(h^{-1} \cdot (s^{-1} \cdot v)) = h^{-1} \cdot \phi(s^{-1} \cdot v)$, so $K(\phi)(v)$ satisfies the H -covariance property and, since it is Borel, it is in $I_H^G(U)$. Now

$$(K(\phi))(t \cdot v)(s) = \phi(s^{-1} \cdot (tv)) = \phi((t^{-1}s)^{-1} \cdot v) = (K\phi)(v)(t^{-1}s)$$

so that $K(\phi)$ intertwines. Finally $\phi \rightarrow K(\phi)$ is an injection since if $K(\phi) = K(\phi')$, then $\phi(s \cdot v) = \phi'(s \cdot v)$ for almost all s for each v . By continuity this is true for all s and hence $\phi(v) = \phi'(v)$.

To complete the proof we must show that K is onto, so let $\psi \in \text{Hom}_G(V, I_H^G(U))$. We have seen in [14, Proposition 17] that $I_H^G(U)$ as a topological group is the same as $U(G/H, U)$ and as $\psi \in \underline{Z}^1(V, U(G/H, U))$, Theorem 2 provides us with a Borel function $x \rightarrow p(x)$ from G/H into $\underline{Z}^1(V, U)$ such that $(\psi(v))(c(x)) = (p(x))(v)$ a.e. where c is the Borel cross section of G/H into G defining the isomorphism of $I_H^G(U)$ with $U(G/H, U)$. Then since $\psi(v)$ is H -covariant we find that $\psi(v)(c(x)h) = h^{-1} \cdot ((p(x))(v))$, and if we write $(P(c(x)h))(v) = h^{-1} \cdot ((p(x))(v))$, then $t \rightarrow P(t)$ is a Borel map of G into $\text{Hom}(V, U)$ such that $\psi(v)(t) = P(t)(v)$ for $v \in V$. We now use the fact that ψ is a G -equivariant map which says that $P(t)(s \cdot v) = \psi(s \cdot v)(t) = P(s^{-1}t)(v)$ holds for almost all t for each s and v . By Fubini we can exclude a null set $N \subset G \times V$ so that if $(t, v) \notin N$, $P(t)(s \cdot v) = P(s^{-1}t)(v) = \psi(s \cdot v)(t)$ holds for almost all s . Now again by Fubini we can select a t so that $(t, v) \notin N$ for almost all v , and so for such a t , $P(t)(s \cdot v) = P(s^{-1}t)(v)$ holds for almost all s and almost all v . But $P(t)(s \cdot v)$ is a continuous function of s for each fixed t and v and so $P(s^{-1}t)(v)$ is equal almost everywhere to a continuous function for almost all v . However, this is additive in v so it is true for all v . In other words, we may modify $P(\cdot)$ on a fixed null set so that $P(s)(v)$ becomes continuous in s , and we assume this done. Then $P(t)(s \cdot v) = P(s^{-1}t)(v)$ still holds for almost all t for each s and v , but now by continuity, it holds for all t . We may then put $s = t$ and conclude that $P(s)(s \cdot v) = P(e)(v)$ and since for some s , $P(s)$ is a continuous homomorphism of V into U , it follows that $\phi = P(e)$ is also. Then $P(s)(w) = \phi(s^{-1} \cdot w)$ and so $\psi(w)(s) = \phi(s^{-1} \cdot w)$ and $\psi = K(\phi)$. Thus the map K is onto as desired, and we are finished.

It is, of course, undesirable to have to restrict V in the above theorem to be locally compact, but one very easily produces counterexamples when this hypothesis is dropped so this is about as well as one can do. Incidentally, we note a somewhat interesting sidelight of the above proof and that is that $(K\phi)(v)(s) = \phi(s^{-1} \cdot v)$ when viewed as an element of $I_H^G(U)$ is a continuous func-

tion on group G . So we have the following fact.

PROPOSITION 1. *Any locally compact G -submodule M of $\Gamma_H^G(U)$ (or indeed any G -submodule which is the continuous image of a locally compact G -module) necessarily consists of continuous functions in the sense that each class $\hat{f} \in M$ contains a (unique) continuous function.*

Let us proceed with other applications of Theorem 1; one of the simplest being a vanishing theorem, namely if $H^n(G, A) = 0$, then it follows that $H^n(G, U(X, A)) = 0$. Let us apply this idea as follows: if B is the torus T , or T^n or T^ω , a countable product of copies of T , and E is any locally compact abelian group satisfying an exact sequence $0 \rightarrow B \rightarrow E \rightarrow G \rightarrow 0$ where G is, of course, abelian, then necessarily E is the product $B \times G$ algebraically and topologically. This is the equivalent of saying that if $\alpha \in H^2(G, B)$ corresponds to an extension E of G by B which is abelian, then $\alpha = 0$. The proof of this statement is routine using Pontrjagin duality, for one converts the above into a dual sequence $0 \rightarrow \hat{G} \rightarrow \hat{E} \rightarrow \hat{B} \rightarrow 0$, and since $\hat{B}$ is free as abelian group and is discrete, $\hat{E} = \hat{G} \times \hat{B}$ and then we dualize again. Now since groups of the form $U(X, T)$ are continuous direct products of circles, we might hope that the same result works for them.

THEOREM 4. *If G is locally compact and if E is an extension of G by a group $U(X, T)$ such that E is abelian, then the extension splits as $U(X, T) \times G$ topologically and algebraically.*

PROOF. Since $U(X, T)$ is divisible as a group it is clear that the extension splits as an abstract group but we want more than that. We first observe that the action of G on $U(X, T)$ in the above extension E is trivial as E is abelian. Now if G is any locally compact group, and A is any abelian G -module with trivial action, and if $a \in Z^2(G, A)$ or $a \in Z^2(G, T)$, then $a'(s, t) = a(t, s)$ is in the same group. Now let $a' = \phi(a)$, $a \in Z^2(G, A)$ ($\phi(a) = a'$ if $a \in Z^2(G, A)$), so that ϕ is an order two automorphism. Since $\phi(a) = a$ for any $a \in B^2(G, A)$ or $a \in \underline{B}^2(G, A)$, $\phi, \underline{\phi}$ defines an order two automorphism again denoted by ϕ ($\underline{\phi}$) on $H^2(G, A)$ ($\underline{H}^2(G, A)$). We need the following fact.

PROPOSITION 2. *$\phi = \underline{\phi}$ and a cohomology class α defines an extension E which is abelian if and only if $\phi(\alpha) = \alpha$.*

PROOF. If $\alpha \in H^2(G, A)$, and a is a cocycle representative, the corresponding group extension may be realized algebraically as $A \times G$ with multiplication given by $(a, g)(a', g') = (a + a' + a(g, g'), gg')$ and so it is immediate that E is abelian if and only if $\phi(a) = a$ or equivalently $\phi(\alpha) = \alpha$. Since the natural map $H^2(G, A) \rightarrow \underline{H}^2(G, A)$ is an isomorphism, it is clear by inspection that $\phi = \underline{\phi}$.

(Note that this implies for instance that an $a \in \underline{Z}^2(G, A)$ such that $a(s, t) = a(t, s)$ for almost all s and t defines an abelian extension.)

Let us now return to the proof of the theorem. The given extension E is represented by a class α in $\underline{H}^2(G, U(X, T))$ with $\phi(\alpha) = \alpha$, and by Theorem 1 there is a Borel map $x \rightarrow \alpha_x$ of G into $\underline{H}^2(G, T)$ corresponding to α . Since $\phi(\alpha) = \alpha$, $\phi(\alpha_x) = \alpha_x$ for almost all x . But then such an α_x defines an abelian extension of G by T which splits as remarked above, and so $\alpha_x = 0$ a.e., and hence $\alpha = 0$ and the extension E splits.

We note that the hypothesis that G be locally compact is essential here, and the result is false in general even for $U(X, T) = T$ for we have already remarked in [14] that if X is nonatomic, then $U(X, T)$ admits no continuous homomorphisms to the circle group T . Then if we embed T into $U(X, T)$ as the constant functions on X we obtain an extension $0 \rightarrow T \rightarrow U(X, T) \rightarrow G \rightarrow 0$ where G is the quotient group which does not split topologically.

Let us give another application of these ideas. Suppose that A is a von Neumann algebra realizable on a separable Hilbert space and let $\text{Aut}(A)$ be the full automorphism group of A , and $\text{Inn}(A)$ the inner automorphisms. The group $\text{Inn}(A)$ is isomorphic to the quotient $U(A)/Z(A)$ where $U(A)$ is the unitary group in A , and $Z(A)$, the unitary operators in the center Z of A . We give this the quotient topology starting from the strong topology on $U(A)$ in which it is polonais. Now $\text{Aut}(A)$ also has a natural topology given by the pointwise convergence on the predual A_* of A , in which it is also polonais, and the natural injection $i: \text{Inn}(A) \rightarrow \text{Aut}(A)$ is continuous and hence also a Borel isomorphism. It follows that if G is locally compact separable or even polonais and if ρ is a homomorphism of G into $\text{Inn}(A)$, then ρ is continuous as a map into $\text{Inn}(A)$ if and only if $i \circ \rho$ is continuous as a map into $\text{Aut}(A)$. Therefore there is no ambiguity about what we mean by a (strongly) continuous representation of G into $\text{Inn}(A)$, the inner automorphisms of G . The problem that arises about such representations ρ is whether or not they are implementable in the sense that there exists a continuous unitary representation π of G with $\pi(g) \in U(A)$ and $\rho(g)(a) = \pi(g)a\pi(g)^{-1}$.

THEOREM 5. *Suppose G is locally compact separable with $H^2(G, T) = 0$. Then any (strongly) continuous representation of G by inner automorphisms of A as above is implementable.*

PROOF. We consider the group extension $1 \rightarrow Z(A) \rightarrow U(A) \rightarrow I(A) \rightarrow 1$ and the homomorphism ρ of G into $I(A)$ gives us a pull back group extension $1 \rightarrow Z(A) \rightarrow E \rightarrow G \rightarrow 1$.

This could be equivalently defined by taking a Borel cross section s of $I(A)$ into $U(A)$ and defining a Borel two cocycle $a'(u, v) = s(u)s(v)s(uv)^{-1}$ of $I(A)$

with values in $Z(A)$. Then $a(g, h) = a'(\rho(u), \rho(v))$ is an element of $Z^2(G, Z(A))$ which defines the extension above. Then if α is the class of a , the point is that the vanishing of α , or the splitting of the above extension is clearly necessary and sufficient for the implementability of ρ . Now in view of our assumptions on A , $Z(A)$ is isomorphic to a group $U(X, T)$ so that $\alpha \in H^2(G, U(X, T))$, which by Theorem 1 is $U(X, H^2(G, T)) = (0)$, by hypothesis. So $\alpha = 0$ and ρ is implementable.

Of special interest is the case $G = \mathbf{R}$ where $H^2(\mathbf{R}, T) = (0)$ is well known. See [7].

COROLLARY. *Any (strongly) continuous one parameter group of inner automorphisms of such an A is implementable.*

We note that for the Poincaré group P , which has such representations ρ of interest, that $H^2(P, T) = (0)$ so that any such ρ is implementable. L. Michel [9] has obtained this result by direct methods sometime ago. We can also find an equivalent statement of Theorem 5 in general with no mention of cohomology.

THEOREM 5'. *Let G be separable locally compact and suppose that any continuous representation of G by (inner) automorphisms of the algebra $B(H)$ of all bounded operators is implementable. Then any continuous representation of G by inner automorphisms of A is implementable.*

PROOF. We note that $H^2(G, T) \neq 0$ is the same as the existence of a non-trivial projective representation, on H and this is exactly the same as a nonimplementable representation of G by automorphisms of $B(H)$.

3. We now want to turn attention to the question of a more detailed study of the topology on $\underline{H}^2(G, A)$ and, in particular, the comparison of this topology with the splitting group topology introduced in [11]. Suppose that $A \in \mathcal{P}(G)$; we shall say the extension $1 \rightarrow B \rightarrow E \rightarrow G \rightarrow 1$ is a splitting group for A if (1) B (and hence also E) are locally compact and, (2) if we let E operate on A by means of the quotient map to G so that B operates trivially, then in the restriction-inflation sequence [10] arising from the spectral sequence

$$0 \rightarrow H^1(G, A) \rightarrow H^1(E, A) \rightarrow H^1(B, A)^G \xrightarrow{tg} H^2(G, A) \xrightarrow{i} H^2(E, A)$$

the transgression homomorphism from $H^1(B, A)^G$ to $H^2(G, A)$ is surjective. Note that $H^1(B, A) = \text{Hom}(B, A)$ is Hausdorff so that Theorem 1.1 of [10] is indeed applicable. Note also that tg is onto if and only if the inflation map i is zero. Now $\text{Hom}(B, A)^G$ is clearly polonais and the kernel of the map tg is the image of the restriction map r from $H^1(E, A)$ which in turn is the quotient of $\underline{Z}^1(E, A)$ by the projection map p . Thus we have $\underline{Z}^1(E, A) \xrightarrow{r \circ p} \text{Hom}(B, A)^G \rightarrow H^2(G, A) \rightarrow 0$ and we can give $H^2(G, A)$ the quotient topology from

$\text{Hom}(B, A)^G$. In fact this is a pseudo-polonais topology and could be represented by the triple $(\underline{Z}^1(E, A), \text{Hom}(B, A)^G, r \circ p)$, and as in [11] we call this the splitting group topology J_s on $H^2(G, A)$. The terminology implies uniqueness (independence of E) which we shall establish presently. But the point is that $H^2(G, A)$ already has a topology from the isomorphism with $\underline{H}^2(G, A)$ which we call J_m (for convergence in measure). Moreover we define a third topology J_c by giving $Z^2(G, A)$ the compact open topology and then taking the quotient topology on $H^2(G, A)$.

PROPOSITION 3. *We have inclusions $J_s \supset J_c \supset J_m$.*

PROOF. That $J_c \supset J_m$ is clear since uniform convergence on compact sets implies convergence in measure. On the other hand, to see that $J_s \supset J_c$ we choose a Borel cross section s of G into E such that $s(K)$ has compact closure in E whenever K is compact in G . (This is always possible by [8, Lemma 1.1].) Then if a is the cocycle in $Z^2(G, B)$ defined by this cross section, a maps compact sets in $G \times G$ into compact sets in B . Now to prove our assertion, we must show that if $\lambda_n \rightarrow \lambda$ in $\text{Hom}(B, A)^G$, then $tg(\lambda_n) \rightarrow tg(\lambda)$ in the topology J_c , or, equivalently, that we may find cocycle representatives b_n of $tg(\lambda_n)$ and b of $tg(\lambda)$, such that $b_n \rightarrow b$ uniformly on compact sets of $G \times G$. In fact, let $b_n(s, t) = \lambda_n(a(s, t))$ and $b(s, t) = \lambda(a(s, t))$; these are cocycle representatives by definition of the transgression map tg , and as $\lambda_n \rightarrow \lambda$ uniformly on compact subsets of B , it follows by the properties of a that $b_n \rightarrow b$ uniformly on compact subsets of $G \times G$. This completes the proof.

Our main result ties all of these topologies together and also shows that J_s when it exists is independent of the choice of splitting group.

THEOREM 6. *If the module A has a splitting group, then the topologies J_s, J_c, J_m on $H^2(G, A)$ all coincide, and hence J_s is independent of which splitting group is used.*

PROOF. In view of the proposition it suffices to show that $J_s \subset J_m$ or that convergence in measure in $\underline{H}^2(G, A)$ implies convergence in J_s defined by a splitting group, say E . We shall do this by constructing a map from $\underline{Z}^2(G, A)$ into $\text{Hom}(B, A)^G$ which on cohomology inverts the transgression map. We first embed $\underline{Z}^2(G, A)$ into $\underline{Z}^2(E, A)$ via the inflation map i as the closed subgroup $\underline{C}$ of the latter group of functions constant almost everywhere on cosets of B . Since the inflation map $i: H^2(G, A) \rightarrow H^2(E, A)$ is the zero map, we see that $\underline{C} \subset \underline{B}^2(E, A)$. Now if δ is the coboundary map, $(\delta)^{-1}(\underline{C})$ is a closed subgroup $\underline{D}$ of $\underline{C}^1(E, A)$. Furthermore we may characterize elements $d \in \underline{D}$ as those such that $\delta(d)$ is constant a.e. on cosets of B . Now δ is of course continuous and if $\underline{M}$ is its kernel, $\underline{D}/\underline{M}$ has a continuous bijection onto $\underline{C}$ which is therefore a Borel iso-

morphism, and it follows that there is a Borel cross section say t from $\underline{C}$ to $\underline{D}$ with $\delta \circ t = \text{id}$.

We now investigate $\underline{D}$ more closely; for $d \in \underline{D}$, $\delta(d)(s, t) = \delta(d)(s, tb)$ holds for almost all triples (s, t, b) in $E \times E \times B$. When we write this out and rearrange terms and substitute $u = st$, $v = t$, we see that $u^{-1} \cdot (d(ub) - d(u)) = v^{-1} \cdot (d(vb) - d(v))$ for almost all triples (u, v, b) in $E \times E \times B$. Consequently, the left-hand side cannot depend on u , and is therefore equal to a function $c_1(b)$ for almost all pairs (u, b) and, similarly, the right-hand side must be the same function. Thus $d(ub) = d(u) + u \cdot c_1(b)$ for almost all pairs (u, b) . On the other hand, the same argument applied to $\delta(d)(bs, t) = \delta(d)(s, t)$ yields an equation $d(bu) = c_2(b) + d(u)$ for almost all pairs. Now

$$\begin{aligned} c_2(b_1 b_2) &= d(b_1 b_2 u) - d(u) = (d(b_1 b_2 u) - d(b_2 u)) \\ &\quad + (d(b_2 u) - d(u)) = c_2(b_1) + c_2(b_2) \end{aligned}$$

is valid for almost all (u, b_1, b_2) , and hence $c_2(b_1 b_2) = c_2(b_1) + c_2(b_2)$ for almost all pairs. By [14, Theorem 3] there is a continuous homomorphism which we also denote by c_2 of B into A which agrees with c_2 almost everywhere. It is easy to see then that we may modify the function d on a null set, and hence make no essential change, so that $d(bu) = c_2(b) + d(u)$ holds for all pairs (b, u) . Thus d is a continuous function on each coset of B . Now finally we can argue in the same way that $c_1(b_1 b_2) = c_1(b_1) + c_1(b_2)$ for almost all pairs (note that B operates trivially on A). Then we use Theorem 3 of [14] again to produce a continuous homomorphism again denoted by c_1 agreeing with the old c_1 almost everywhere. Then we have $d(ub) = d(u) + u \cdot c_1(b)$ holding for almost all pairs. But we have arranged for $d(u)$ to be continuous on each coset which implies that $d(ub)$ is continuous on G and the right-hand side is also continuous; hence for almost all u , equality holds for all b . Then for such a u , $d(ub) = d(ubu^{-1}u) = d(u) + c_2(ubu^{-1}) = d(u) + u \cdot c_1(b)$ and so $c_2(ubu^{-1}) = u \cdot c_1(b)$ holds for almost all u . But both sides are continuous in u and this must hold for all u . For $u = e$, we find $c_2 = c_1$ and so we drop the subscripts, and call it c ; we have $c(ubu^{-1}) = u \cdot c(b)$ so that $c \in \text{Hom}(B, A)^G$. Moreover it follows readily that $d(ub) = d(u) + u \cdot c(b)$ for all u and b so in fact $c(b) = d(b)$.

Now let s be a Borel cross section of G into E as in the lemma with $s(e) = e$. Let us define $d_1(b \cdot s(g)) = d(s(g))$ so that d_1 is Borel function constant on cosets of B . It is therefore clearly a member of the group $\underline{D}$. Now let us also define $\phi(c)(b \cdot s(g)) = c(b)$. Then a simple calculation shows that $\phi(c) \in \underline{D}$ and that $d = \phi(c) + d_2$. Clearly $\phi(c)$ can be defined for any $c \in \text{Hom}(B, A)^G$ and $c \rightarrow \phi(c)$ is clearly a continuous injective homomorphism of this group into $\underline{D}$. Finally let $\underline{N}$ be the subgroup of $\underline{D}$ consisting of functions constant on cosets of B , and ψ be the map from $\text{Hom}(B, A)^G \times \underline{N}$ into $\underline{D}$ given by $\psi(c, d_2) = \phi(c) +$

d_2 . Our result above is precisely that ψ is onto, and it is clear that it is also continuous and injective. Then by the closed graph theorem, it is a homeomorphism. Let p denote the composition of the projection to the first factor with the inverse of ψ so that $p(d) = c$ if $d = \phi(c) + d_2$ as above. Then of course p is a continuous homomorphism from $\underline{D}$ onto $\text{Hom}(B, A)^G$.

Recall finally that i embedded $\underline{Z}^2(G, A)$ into $\underline{Z}^2(E, A)$ as a group $\underline{C}$, and that t is some Borel cross section of $\underline{C}$ back to $\underline{D}$ inverting δ , and we consider the composition $\theta = p \circ t \circ i$ of $\underline{Z}^2(G, A)$ into $\text{Hom}(B, A)^G = \underline{H}^1(B, A)^G$ which is a Borel map. We want to show that the identity map from $\underline{H}^2(G, A) = H^2(G, A)$ in the topology J_m to itself with the topology J_s is continuous. To do this it is enough to show that the projection into $H^2(G, A)$ mod the closure of the identity $(\bar{e})_s$ in the topology J_s is continuous. However, from the definition of J_s , $H^2(G, A)/(\bar{e})_s$ can be identified with $H^1(B, A)^G$ modulo the closure of the kernel of the transgression homomorphism, which we denote by K . Let q denote the projection into this quotient, and we see that the map $q \circ \theta$, θ as above, from $\underline{Z}^2(G, A)$ into $H^1(B, A)^G/K$ is now a homomorphism and as it is Borel, it is necessarily continuous. Finally $q \circ \theta$ vanishes on $\underline{B}^2(B, A)$ and, hence, defines a continuous map of $\underline{H}^2(G, A)$ into $H^1(B, A)^G/K$, which one can see from the algebra is precisely the map we need to know is continuous. This completes the proof of the theorem.

We might remark that it is somewhat surprising that J_c , the compact open topology on Borel cycles turns out in many cases to be quite reasonable on the quotient $H^2(G, A)$. The example discussed in [11, p. 84] where the splitting group topology on $H^2(G^*, T)$ is the reals mod the rationals R/Q gives the same result for the topology J_m . We might add that the previous theorem provides the missing converse of Theorem 2.5 of [11] and much more.

Let us turn for a moment to the question of when our cohomology groups are Hausdorff. In general it is hard to say when they are Hausdorff and we have only fragmentary results. One such is the following. (See also Theorem 13.)

THEOREM 7. *If G is abelian, then $\underline{H}^2(G, T)$ is Hausdorff.*

PROOF. We consider the mapping ϕ of $\underline{Z}^2(G, T)$ to itself, introduced in Theorem 3, defined by $\phi(a)(s, t) = a(t, s)$, and we observed there that $\phi(a) = a$ if and only if the corresponding extension E was abelian. But with T as coefficient group an extension E is trivial if and only if it is abelian. Thus $a \in \underline{B}^2(G, T)$ if and only if $\phi(a) = a$, but ϕ is clearly a continuous map and so $\underline{B}^2(G, T)$ is closed as desired.

As we noted earlier, Johnson [4] has introduced cohomology groups associated to a representation of G by bounded operators on a Banach space B . We want to discuss the relation between his theory and ours. Thus suppose that

$A = B$ is a Banach space and that we are given a homomorphism ρ of G into the invertible operators on B with $|\rho(g)| \leq K$. If B is norm separable there is little question as to what continuity conditions to impose—namely that B be a G -module in the sense we are using. On the other hand, there are many interesting nonseparable B where we should like the theorem to apply. This problem and others are taken care of by assuming that we have a “separable weak topology” J on B by which we understand a locally convex separated topology on B usually weaker than the norm topology such that the unit ball B_1 is J -compact and metrizable. We assume that each $\rho(g)$ is J continuous and that $g \rightarrow \rho(g)(v)$ is J continuous in which case we say that (B, J) is a G -module. It is very easy to see by the same kind of arguments used in [14, Proposition 11] that $(g, v) \rightarrow \rho(g)(v)$ is jointly continuous from $G \times B_r \rightarrow B$ where B_r is the ball of radius r , and where J is taken as the topology on B_r and B . If B itself is separable it is immediate that the Borel structure on B_r generated by J is the same as that generated by the norm topology. Then $(g, v) \rightarrow \rho(g)(v)$ is jointly Borel from $G \times B_r$ into B , and hence immediately jointly Borel from $G \times B$ into B , and then by Proposition 11 it is jointly continuous and, hence, B is a G -module. Conversely, if B is a G -module, and $\rho(g)$ is J continuous for each g , then one easily deduces that $g \rightarrow \rho(g)(v)$ is continuous from G to B with J as topology. Hence our assumptions on the action are equivalent to B being a G -module if B is separable, and we do include a large number of interesting examples for which B is nonseparable. It goes without saying of course that the most common examples of weak topology arise when B is the dual of a separable Banach space B_* and we use the weak*-topology on B in that case.

Let us agree to say that a bounded function from G into B is weakly Borel if it is Borel with respect to the σ -field generated by J . In the following theorem, $Z^1(G, B)$ is, strictly speaking, undefined unless B is separable, but as one can see that does not really matter as long as it consists of functions satisfying $f(st) = \rho(s)f(t) + f(s)$, for what the theorem is describing is just $B^1(G, B)$, which is by definition all functions of the form $f(g) = \rho(g)b - b$ for some $b \in B$.

Johnson establishes a fundamental vanishing theorem which in our context becomes the following. We include the proof since it is quite short.

THEOREM (JOHNSON). *If G is amenable and operates by ρ on B by a uniformly bounded representation and if B admits a separable weak topology J , so that (B, J) is a G -module, then $B^1(G, B)$ is the set of bounded weakly Borel functions in $Z^1(G, B)$.*

PROOF. We can of course use the invariant mean on G to renorm B up to equivalence so that $\rho(g)$ is an isometry but that is not crucial. In any case if $f \in \underline{B}^1(G, B)$, $f(s) = \rho(s)(b) - b$ is clearly a bounded weakly continuous function.

On the other hand, if f is a bounded one cocycle f takes values in a weakly compact and metrizable set. If f is weakly Borel it follows by standard arguments that it is also weakly continuous. We then form the function $h(s) = \rho(s^{-1})(f(s))$ which is also bounded and then form its right invariant mean $b = M(h)$ which is possible since the norm balls in B are weakly compact. Then we compute that

$$\begin{aligned}\rho(t)b - b &= \rho(t)(M(h)) - M(h) = M(\rho(t)h) - M(h) \\ &= M_s(\rho(ts^{-1})f(s)) - M(h)\end{aligned}$$

and since $\rho(ts^{-1})f(s) = f(t) - f(ts^{-1})$ we find that the above is equal to

$$M_s(f(t) - f(ts^{-1})) - M(h) = f(t) - M_s(f(s^{-1})) - M_s(\rho(s^{-1})f(s)),$$

but now $\rho(s^{-1})f(s) + f(s^{-1}) = 0$ so that the last two terms cancel and we have $\rho(t)(b) - b = f(t)$ is a coboundary as desired.

The hypotheses of the theorem regarding J are satisfied if, for instance, B is the dual of some separable Banach space B_* and we take J to be the weak*-topology. Our assumption on $\rho(g)$ is just that it is the dual of some operator $\lambda(g)$ on B_* which is a natural assumption. This theorem extends results of Browder [2] for we simply take G to be the integers acting by isometries so that we are considering a single invertible isometry T of B . All Borel conditions evaporate and a cocycle f is entirely determined by its value at 1, $x = f(1)$, and the value $f(n)$ is simply $\sum_{i=0}^{n-1} T^i(x)$. That f is a coboundary is the same as saying that $x = Ty - y$ has a solution y ; our result above simply states under the existence of an appropriate weak topology that $x = Ty - y$ if and only if the norms of sums $|\sum_{i=0}^n T^i(x)|$ are uniformly bounded in n . Our interest in this question was in part motivated also by a question of I. Segal corresponding to the case $G = \mathbb{R}$, and B a von Neumann algebra with R acting by $*$ automorphisms. Here we have a natural predual B_* and the theorem applies.

Browder also observes that there are some situations where the theorem is true even without the existence of J . For instance, if M is a compact metric space, if φ is a minimal homeomorphism of M , and if U is the induced operator on $C(M)$, all continuous complex valued functions on M , the result characterizing solvability of $f = Ug - g$ still holds. It might be worthwhile to point out that a very slight extension to the nonminimal case of this result fails. Let us consider $M = T \times T$, T the circle, and let Z act by $n \cdot (u, v) = (s^n u, v)$ where s is not of finite order. This is an isometric action which decomposes trivially into disjoint minimal sets which in this case even give a fibration of M . If U is the induced operator on $C(M)$, we claim the theorem is false and we can produce $f \in C(M)$ with $|\sum_{i=0}^n U^i f|$ uniformly bounded but such that $f = Ug - g$ has no solution in $C(M)$. Of course it follows from the theorem that it has a solution in $L_\infty(M)$. If

g were a solution to such an equation we note that $g(\cdot, v)$ must for each v be unique up to an additive constant since only the constant functions are invariant under translation by s . Therefore if we have a solution we can normalize it by subtracting the function $(u, v) \rightarrow g(1, v)$ so that the result satisfies $g(1, v) = 0$ in which case the solution g is unique if it exists. Now if U is the linear map on $C(T)$ given by $(Uf)(u) = f(su)$, then $1 - U$, even when restricted to the subspace of those g such that $g(1) = 0$, has a nonclosed range and is one-one. Its inverse must be discontinuous and so there is a sequence g_n with $g_n(1) = 0$ and with $|g_n| = 1$ but with $|f_n| \rightarrow 0$ where $f_n = (U - 1)g_n$. Then we define a function g on $T \times T$ by

$$g(u, \exp(2\pi i(t(n^{-1}) + (1-t)(n+1)^{-1}))) = tg_n + (1-t)g_{n+1}$$

for $0 \leq t \leq 1$, and $g(u, 0) = 0$, and $g(u, \exp(2\pi i(-s))) = g(u, \exp(2\pi is))$. The function g is clearly continuous at every point of $T \times T$ except possibly at points $(u, 1)$. Moreover $g(1, v) = 0$. We let $f(u, v) = g(su, v) - g(u, v)$. Then f is given by the same kind of formula as g and since $|f_n| \rightarrow 0$ it is clear that f is, in fact, continuous everywhere. By our remarks above, if $(U - 1)h = f$ is to have a solution h , then the function g above has to be continuous, but it is clearly not continuous, for as $|g_n| = 1$ there is a point u_n where $|g_n(u_n)| = 1$, and we may take a subsequence $n(m)$ such that $u_{n(m)}$ converges to u_0 and $g_{n(m)} \rightarrow \alpha$, $|\alpha| = 1$. Then $g(u_{n(m)}, \exp(2\pi i(n(m))^{-1})) \rightarrow \alpha$ but $g(u_0, 1) = 0$ so g is discontinuous.

One rather immediate application of Johnson's theorem on bounded co-cycles concerns the existence of invariant measures for group actions. Specifically, assume that a locally compact separable group G acts as a Borel transformation group on a standard Borel space X , leaving a measure μ quasi-invariant (cf. [1]). Then if for each $g \in G$, we let $r(g)$ be the logarithm of the Radon-Nikodým derivative of the transformed measure $g \cdot \mu$ with respect to μ , it may be verified trivially that $r(g)$ is a one cocycle in $Z^1(G, U(X, R))$ where G operates as usual on the module in question. Moreover, if we replace μ by an equivalent measure ν given by a Radon-Nikodým derivative f , then the corresponding cocycle changes by a coboundary, namely $\delta_0(\log f)$, where $\log f$ is viewed as an element of $U(X, R)$. Thus the cohomology class of r , $[r]$, is uniquely determined, and it is clear that $[r] = 0$ if and only if the group action admits an invariant measure ν equivalent to the given measure μ . The vanishing theorem has the following immediate consequence (cf. [6, p. 99]).

THEOREM 8. *Let G , X , μ , and r be as above, and suppose that all the $r(g)$ are essentially bounded functions on X with a bound independent of g . Then if G is amenable, there is a G -invariant measure ν equivalent to μ .*

PROOF. We use the Banach space $B = L_\infty(X)$ and use the weak*-topology

from $L_1(X)$. It is clear that the hypotheses are verified and we conclude that $[r]$ viewed as an element of $H^1(X, B)$ is zero and hence that $[r] = 0$ in $H^1(X, U(X, R))$ since $B \subset U(X, R)$. Note that as $r = \delta_0(\log f)$ with $\log(f)$ bounded, we can even assert that there exists an invariant measure ν such that $d\nu/d\mu$ is bounded above and below (in fact by a bound determinable from the original bound on the sup norms of $r(g)$).

This theorem would appear to depend in an essential way on the hypothesis that G is amenable; however, by using totally different techniques, we are now able to establish this same theorem for arbitrary G . This result will appear in a subsequent paper.

4. We shall now turn our attention to some questions and unanswered problems arising in [13]. We recall that a locally compact G is said to be "simply connected" in the algebraic sense if every extension $1 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1$ with A a locally compact G -module with trivial action splits and splits uniquely; that is, there is one and only one continuous homomorphism of G into E inverting the projection. We observed that this condition is equivalent to $H^1(G, B) = H^2(G, B) = (0)$ for every locally compact trivial G -module. The first condition is equivalent to $H^1(G, T) = \text{Hom}(G, T) = (0)$ which in turn is the same as $[G, G]$, the commutator subgroup being dense. We observed that the second is equivalent to $H^2(G, T) = (0)$ and $H^2(G, D) = 0$ for every discrete group D , and raised the question whether indeed the second condition (which might be hard to verify) is superfluous. We can show now that it is and even establish somewhat more. Let us recall that a polonais group was said to be unitary if it can be realized as a closed subgroup of the unitary operators on a separable Hilbert space. We noted that all locally compact separable groups are unitary as the regular representation provides such an embedding. It might be useful to also point out that if A is a polonais G -module which is unitary, we can find a realization of A as a unitary group on some H together with a unitary representation π of G on H such that $g \cdot a = \pi(g)a\pi(g)^{-1}$; that is the action of G can be implemented by a unitary representation. To see this, note that if A is unitary, then $I(A)$ is also unitary and that the action of G on $I(A)$, which is just translation, is implementable by a unitary representation. Then we note that A as G -module can be embedded as a sub- G -module of $I(A)$ and we are done. We return to the situation at hand regarding simply connected groups.

PROPOSITION 4. *If $H^1(G, T) = H^2(G, T) = (0)$, then G is simply connected. Moreover $H^2(G, A) = 0$ for any unitary G -module with trivial action.*

PROOF. The first statement clearly follows from the second, and so we have only to prove it. As $A \subset U(H)$ for some separable Hilbert space H , let A

be the von Neumann algebra generated by A and let U be its unitary group. Now A is clearly abelian, and so U has the form $U(X, T)$ and hence we have an embedding $A \rightarrow U(X, T)$ of A as a closed subgroup. Let B denote the quotient so that $0 \rightarrow A \rightarrow U(X, T) \rightarrow B \rightarrow 0$ is an exact sequence of (trivial) G -modules. Now the long exact sequence of cohomology together with the observation that $H^1(G, B) = \text{Hom}(G, B) = (0)$ as $[G, G]$ is dense implies that the induced map $H^2(G, A) \rightarrow H^2(G, U(X, T))$ is injective. But the latter group is by Theorem 1 equal to $U(X, H^2(G, T))$ which is zero by assumption. Thus $H^2(G, A) = (0)$ as desired and we are done.

Recall that a surjective continuous homomorphism p of locally compact groups $E \rightarrow G$ was said to be a covering map if $[E, E]$ is dense in E (i.e. $H^1(E, T) = 0$) and if the kernel of p is central in E . Then it follows that E is an extension of G by the trivial G -module $\ker(p)$ and of course $[G, G]$ is necessarily dense in G . In [13], we discussed the question of finding for any G with $[G, G]$ dense in G , an E with a covering map p onto G such that E is simply connected. If such an E exists it is unique and indeed the whole extension of G by $\ker(p)$ is essentially unique. In analogy with the situation in topology we say that E is (the) simply connected covering group of G , and the kernel of p , which we denote by $\pi_1(G)$, is called the fundamental group of G . It is an abelian locally compact group (and there are cases when the topology is nontrivial). The justification for this terminology comes from the fact that if G is a semi-simple Lie group, it has a universal covering group in our sense which coincides with the topologically defined universal covering group, and our $\pi_1(G)$ is discrete and coincides with the usual fundamental group.

Suppose for the moment that G does have a universal covering group E ; then the restriction inflation sequence reads

$$(0) = H^1(E, A) \rightarrow H^1(\pi_1(G), A) \rightarrow H^2(G, A) \rightarrow H^2(E, A) = (0)$$

for any unitary G -module with trivial action. Then in terms of the previous section we have the following consequence of Theorem 6.

PROPOSITION 5. *If G has a universal covering group E , then E is a splitting group for any unitary G -module with trivial action. Moreover the splitting group topology is Hausdorff and the transgression map $H^1(\pi_1(G), A) = \text{Hom}(\pi_1(G), A) \rightarrow \underline{H}^2(G, A)$ is a homeomorphism from the compact open topology on the first group to the topology of convergence in measure on the second group.*

We have already mentioned in [11, p. 84] and earlier in this paper an example of a group G^* for which $[G, G]$ is dense in G but with $H^2(G, T)$ topologically isomorphic to R/Q in the splitting group topology J_s and hence also in J_m . Specifically this group is obtained by looking at the universal covering group

G of $SL_2(R)$ which is an extension of $SL_2(R)$ by the integers (natural numbers) Z , and then replacing Z by Z^* , the compactification of Z with respect to all subgroups of finite index and taking the image of the cocycle defining G in $H^2(SL_2(R), Z^*)$. It is immediate that $[G^*, G^*]$ is dense in G^* but not equal to G^* and the proposition above implies that G^* cannot have a universal covering group. Thus the density of $[G, G]$ in G is not sufficient for existence of a universal covering group. However, we can still ask if $[G, G] = G$ is a sufficient condition. Even this turns out to be false as we shall see momentarily, although there is a weaker replacement for it. Before we proceed to that, however, we shall give proofs of two affirmative results which were only stated in [13] with a promise that their proofs would appear in a subsequent paper (this one). These results are as follows.

THEOREM 9. *If $G = [G, G]$ and if G is almost connected, that is, G/G_0 is compact, where G_0 is the identity component, then G has a universal covering group.*

THEOREM 10. *If G is a connected Lie group with $[G, G] = G$, then its universal covering group is again a connected Lie group with $E = [E, E]$, and $\pi_1(G) \simeq \pi_1^{\text{top}}(G) \oplus H^2(\mathfrak{g}, R)^*$ where $\pi_1^{\text{top}}(G)$ is the usual topological fundamental group and where $H^2(\mathfrak{g}, R)$ is the two-dimensional Lie algebra cohomology vector group.*

PROOF OF THEOREM 9. We have shown in [11] that there is a group F which is a central extension of G by some locally compact group B such that F is a splitting group for the circle group T as trivial G -module. Now let E be the closure of the commutator subgroup of F . If p is the projection from F to G , it follows that $p(E) = G$ since $G = [G, G]$. (Note that this might fail if $[G, G]$ is only dense in G .) We set $A = B \cap E$, and E is a central extension of G by A . Moreover as $F = E \cdot B$ where B is central, $[F, F] = [E, E]$ is dense in E , so $H^1(E, T) = 0$. By Proposition 4, it remains to show that $H^2(E, T) = 0$.

First of all the inflation map i_E from $H^2(G, T)$ to $H^2(E, T)$ can be obtained as the composition $r' \circ i_F$ where i_F is the inflation map to $H^2(F, T)$ and r' is restriction to $H^2(E, T)$. Now $i_F = 0$ is equivalent to the fact that F is a splitting group, and it follows that $i_E = 0$. Moreover, we claim that the restriction map r from $H^2(E, T)$ to $H^2(A, T)$ is also the zero map. This follows just as [13] from the observation that if $a \in Z^2(E, T)$, then $b(x, y) = a(x, y)a(y, x)^{-1}$ for $x \in E$ and $y \in A$ is the commutator $[\dot{x}, \dot{y}]$ in the group extension H of E by T defined by a and where $\dot{x}$ and $\dot{y}$ are any elements of H projecting onto x and y . Since $[E, E]$ is dense in E , it follows that $b(x, y) = 0$ since b is continuous and bilinear in x and y . Thus for $x, y \in A$, $a(x, y) = a(y, x)$ and the restriction of a

to A is symmetric, and we have already observed that then the class of a is trivial.

Finally the spectral sequence of the group extension G by A gives us an injection of the quotient group $\ker(r)/\text{Im}(i_E)$ into $H^1(G, H^1(A, T)) = \text{Hom}(G, \text{Hom}(A, T))$ which is zero since $G = [G, G]$. Thus $\ker(r) = \text{Im}(i_E) = (0)$ and we have just shown that $\ker(r)$ is all of $H^2(E, T)$. This completes the proof of Theorem 9.

PROOF OF THEOREM 10. We know that G has a universal covering group E by the above. Now $G = [G, G]$ implies that $[\mathfrak{G}, \mathfrak{G}] = \mathfrak{G}$ where $\mathfrak{G}$ is the Lie algebra of G and so if $\tilde{G}$ is the ordinary topological covering group of G , $[\tilde{G}, \tilde{G}] = \tilde{G}$ so that $\tilde{G}$ is a covering group in our sense. The universal property of E implies that E is (the universal) covering group of $\tilde{G}$ also. Then we also have an exact sequence for $\pi_1(G)$ as follows:

$$0 \rightarrow \pi_1(\tilde{G}) \rightarrow \pi_1(G) \rightarrow \pi_1^{\text{top}}(G) \rightarrow 0$$

since $\pi_1^{\text{top}}(G)$ is the kernel of the map $\tilde{G} \rightarrow G$. If we can show that $\pi_1(\tilde{G})$ is the vector space it is asserted to be, the sequence above splits topologically since the kernel is divisible and the quotient discrete. To put matters another way, we have shown that the result for G follows if we know it for $\tilde{G}$, and we may and henceforth do assume that $G = \tilde{G}$ is topologically simply connected.

Let $Z^2(\mathfrak{G}, R)$ be the vector space of cocycles in dimension two for the Lie algebra cohomology of $\mathfrak{G}$ with coefficients in R , the real line, and let $B^2(\mathfrak{G}, R)$ be the coboundaries and choose a linear map $t: H^2(\mathfrak{G}, R) \rightarrow Z^2(\mathfrak{G}, R)$ inverting the quotient map. Now for each $X, Y \in \mathfrak{G}$, $\alpha \rightarrow t(\alpha)(X, Y)$ is linear in $\alpha \in H^2(\mathfrak{G}, R)$ and defines an element $\beta(X, Y)$ of $V = H^2(\mathfrak{G}, R)^*$, the linear dual of $H^2(\mathfrak{G}, R)$. It is immediate that β is a two cocycle of $\mathfrak{G}$ with values in V , and we let e be the corresponding central extension $0 \rightarrow V \rightarrow e \rightarrow \mathfrak{G} \rightarrow 0$. One has to verify two facts, namely that $[e, e] = e$ and that the inflation map $H^2(\mathfrak{G}, R) \rightarrow H^2(e, R)$ is the zero map—the latter follows simply because we have constructed e in the way we did, and it is the smallest such e that has this property. Then one observes that $[e, e]$ also has this property so that $e = [e, e]$ as desired. We can now precisely copy the argument in Theorem 9 transcribed into Lie algebra terms and conclude that $H^2(e, R) = 0$.

Now we let E be the topologically simply connected Lie group with Lie algebra e . Then as G is simply connected we find an exact sequence of groups $1 \rightarrow V \rightarrow E \rightarrow G \rightarrow 1$ where we have identified the abelian Lie algebra V with the corresponding simply connected group. Since $[e, e] = e$, we have $[E, E] = E$ and as V is central, E is a covering group of G . But we showed in [13] that for topologically simply connected Lie groups E , $H^2(E, T) \simeq H^2(E, R) \simeq H^2(e, R)$ and the final group is zero. Thus E is simply connected in our sense, and we observe then that $\pi_1(G) = V$ as desired.

We now turn to the question of more general sufficient conditions for the existence of a universal covering group. We note that if such a covering group existed it would follow that the natural topology J_m on $\underline{H}^2(G, T)$ is first of all Hausdorff and secondly is locally compact as it is the Pontrjagin dual of $\pi_1(G)$ by virtue of Proposition 5. We can now establish the converse of this.

THEOREM 11. *Suppose $G = [\overline{G}, \overline{G}]$; then G has a universal covering group E if and only if $\underline{H}^2(G, T)$ is locally compact and Hausdorff in the topology J_m .*

PROOF. We have noted the necessity of the condition. Conversely suppose that $\underline{H}^2(G, T)$ is locally compact, and Hausdorff. Let us consider the group extension $0 \rightarrow \underline{B}^2 \rightarrow \underline{Z}^2 \rightarrow \underline{H}^2$ where we abbreviate $\underline{X}^2 = \underline{X}^2(G, T)$ for $X = B, Z, H$. We note that δ^1 maps $\underline{C}^1(G, T)$ onto $\underline{B}^2$ and has kernel $\underline{Z}^1(G, T)$ which is zero by hypothesis. Now since $\underline{B}^2$ is closed, it follows that δ^1 is, in fact, an isomorphism of topological groups and, in particular, $\underline{B}^2 \simeq \underline{C}^1(G, T)$ is a group of the form $U(G, T)$. Now since the total group $\underline{Z}^2$ is abelian and the quotient $\underline{H}^2$ is locally compact, our splitting theorem (Theorem 3) applies and $\underline{Z}^2 \simeq \underline{B}^2 \times \underline{H}^2$, and let us fix some continuous homomorphism φ of $\underline{H}^2$ into $\underline{Z}^2$ defining this decomposition. Now $a \rightarrow \varphi(a)$ is a continuous homomorphism of $\underline{H}^2$ into $\underline{Z}^2 \subset U(G \times G, T)$. We apply our direct integral theorem (Theorem 1) to see that φ corresponds to an element a of $U(G \times G, \text{Hom}(\underline{H}^2, T))$ which we may verify directly to be an element of the subgroup $\underline{Z}^2(G, \text{Hom}(\underline{H}^2, T))$ since $\varphi(a) \in \underline{Z}^2(G, T)$. Now let $A = \text{Hom}(\underline{H}^2, T)$ and let E be the group extension of G by A defined by the cocycle a constructed above. To complete the proof, all we have to do is show that E is simply connected or, equivalently, that $H^1(E, T) = H^2(E, T) = 0$.

We consider the restriction inflation sequence from the group extension and, in particular, we look at the transgression homomorphism $tg: H^1(A, T) = \text{Hom}(A, T)$ into $H^2(G, T) = \underline{H}^2(G, T) = \underline{H}^2$. By duality $\text{Hom}(A, T) = \text{Hom}(\text{Hom}(\underline{H}^2, T), T) \simeq \underline{H}^2$: Moreover if we look at the direct definition of the transgression map in terms of cocycles (after all it is a special case of the Mackey obstruction map) in [5], we see that tg is in fact the identity map from $\underline{H}^2$ to $\underline{H}^2$. Now as $H^1(G, T) = 0$ is given, the restriction-inflation sequence is

$$0 \rightarrow H^1(E, T) \rightarrow H^1(A, T) \xrightarrow{tg} H^2(G, A) \xrightarrow{i} H^2(E, A).$$

The fact that tg is a bijection tells us two things; first that $H^1(E, T) = 0$ and, secondly, that the inflation map i is the zero map. We then use these two facts as in the proof of Theorem 9 to show that $H^2(E, T) = 0$. This completes the proof.

As an example of this we have

THEOREM 12. *Suppose $[\overline{G}, \overline{G}] = G$ and that $H^2(G, T)$ is countable; then G has a universal covering group and in this case $\pi_1(G)$ is compact.*

PROOF. We observe that $\underline{B}^2(G, T)$ is the continuous one-one image of the polonais group $\underline{C}^1(G, T)/\underline{Z}^1(G, T)$ and so it is a Borel subset. It has countable index in $\underline{Z}^2(G, T)$ and if we could show it was open, then $\underline{H}^2(G, T)$ would be Hausdorff and locally compact and the previous theorem would apply. Moreover $\pi_1(G)$ being the dual group of $\underline{H}^2(G, T)$ would be compact. Thus the proof is reduced to the following fact.

PROPOSITION 6. *If A is polonais and B is a Borel subgroup of countable index, then B is open in A .*

PROOF. Let $X = A/B$ and give this the discrete Borel structure—all subsets are Borel. Then it may be verified that the natural transitive operation of G on X is a Borel transformation group on a standard space X . Moreover the counting measure μ on X is quasi-invariant and so G operates on the measure algebra M of X , but now by Proposition 11 of [14] this action must be continuous. We observe that B is precisely the isotropy group of an atom of M and hence B is closed. Then a category argument shows us immediately that B is open and the proposition and Theorem 12 are proved.

We have a corollary of this as follows:

COROLLARY. *If $[G, G]$ is dense in G and if G is almost connected and if $H^2(G, R) = 0$, then G has a covering group.*

PROOF. For almost connected groups we proved in [11] that a cokernel of the map $H^2(G, R) \rightarrow H^2(G, T)$ was countable, and hence in this case the hypotheses of Theorem 12 are satisfied.

We now consider what happens in general if one assumes the stronger condition that $[G, G] = G$ and is not just dense in G .

THEOREM 13. *If $G = [G, G]$ and if A is any abelian polonais G -module with trivial G -action, then $\underline{H}^2(G, A)$ is Hausdorff.*

PROOF. Let $a_n \in \underline{B}^2(G, A)$ with $a_n \rightarrow a$; we shall show $a \in \underline{B}^2(G, A)$. First we can pass to a subsequence and assume that $a_n \rightarrow a$ almost everywhere and moreover we can modify each a_n and a on a null set so that they are Borel and satisfy the cocycle identity everywhere. Then we construct the semidirect product $H = I(A) \cdot G$, and for each $b \in \underline{Z}^2(G, A)$ we let $E(b)$ be the corresponding group extension of G by A constructed in [14, Theorem 10] as a subgroup of H . Specifically let $T(b, g)(x) = b(g, g^{-1}x)$ so that $T(b, g) \in I(A)$ and let $L(b, g) = (T(b, g), g)$. Then the group $E(b)$ is the group generated by $A \subset I(A)$

as constants, and the elements $L(b, g)$, $g \in G$. Now if b is trivial, the extension splits, so there is a unique continuous homomorphism $M(b)$ of G into $E(b)$ of the form $M(b)(g) = (N(b)(g), g)$. Now the $I(A)$ component $N(b)$ of $M(b)$ is easily seen to be a crossed homomorphism of G into $I(A)$ of the form $N(b)(g) = T(b, g) + A(b, g)$ and we have $M(b)(g) = A(b, g) \cdot L(b, g)$ where $A(b, g)$ is in A (which is in the center of H), and T and L are as above.

Then if $g_1, g_2 \in G$, let $[g_1, g_2]$ be their commutator, and it follows from the observation above that $M(b)([g_1, g_2]) = [L(b, g_1), L(b, g_2)]$. Now if $b = a_n$, we see that by Fubini, $a_n(g, g^{-1}x) \rightarrow a(g, g^{-1}x)$ for almost all x for almost all g , and consequently we may exclude a null set N in g so that if $g \notin N$, $T(a_n, g) \rightarrow T(a, g)$ and $L(a_n, g) \rightarrow L(a, g)$. It follows therefore that $M(a_n)([g_1, g_2])$ has a limit, namely $[L(a, g_1), L(a, g_2)]$ in H provided $g_1, g_2 \notin N$, since each $M(a_n)$ is a homomorphism, it follows that $M(a_n)(s)$ has a limit for all s in the subgroup G' of G generated by all commutators $[g_1, g_2]$, $g_i \notin N$.

PROPOSITION 7. *We have $G' = G$.*

PROOF. A standard formula for commutators gives

$$[g_1 g_2, g_3] = [g_1, [g_2, g_3]] [g_2, g_3] [g_1, g_3].$$

Then as $N^c \cdot N^c = G$, we see that G' contains all commutators $[g, g_3]$ for $g \in G$ and $g_3 \notin N$. A similar argument on the second variable shows that G' contains all commutators and hence is equal to G .

It follows then that the sequence of homomorphisms $M(a_n)$ converges pointwise on G to some limit which is necessarily a homomorphism M of G into $H = I(A) \cdot G$. Moreover, M , being the limit of a sequence of Borel maps, is Borel, and then as usual it follows that M is continuous. Moreover for $g = [g_1, g_2]$, $g_i \notin N$, we have noted that

$$M(a)(g) = \lim M(a_n)(g) = [L(a, g_1), L(a, g_2)] \in E(a),$$

and so $M(a)(g) \in E(a)$ for all $g \in G'$ and, hence, for all $g \in G$. Since the second component of $M(a)(g)$ is necessarily g , as the same was true for each $M(a_n)(g)$, $M(a)$ is a splitting homomorphism for the group $E(a)$ and, hence, $a \in \underline{B}^2(G, A)$ as desired.

We have as an immediate corollary the following fact.

COROLLARY. *If $G = [G, G]$ then G has a universal covering group if and only if $\underline{H}^2(G, T)$ is locally compact.*

The final point that we come to is to give an example of a group G with $[G, G] = G$, so that $\underline{H}^2(G, T)$, which is necessarily Hausdorff, is not locally compact. At the same time we shall resolve negatively another question that was left

open in [13]; more precisely let K be a compact open subgroup of a locally compact group H such that K and H both have fundamental groups. Then there is a continuous map of $\pi_1(K)$ into $\pi_1(H)$ and since $\pi_1(K)$ is compact, the image is closed. The question is whether the image is also open, as it is in a great many special cases. It is true when K is normal in which case the cokernel of the map is the fundamental group $\pi_1(H/K)$ of the discrete quotient group G/K , which is known to be discrete. We shall see that the answer to this question in general is "no", and this example, as we shall see, will lead us naturally to a counterexample to the question of existence of fundamental groups.

Let H_0 be the free abelian group on five generators (x_i) and let A_0 be the alternating group on five symbols acting on H_0 by permuting the x_i . Let H be the invariant subgroup of all elements $\prod x_i^{n_i}$ with $\sum n_i = 0$, and let $G_0 = H \cdot A_0$ be the semidirect product. Note that $y_i = x_i x_{i+1}^{-1}$, $i = 1, 2, 3, 4$, are free generators for H , and if $\sigma \in A_0$,

$$\sigma y_i \sigma^{-1} y_i^{-1} = x_{\sigma(i)} x_{\sigma(i+1)}^{-1} x_{i+1} x_i^{-1}.$$

If $\sigma(i+1) = i+1$, this becomes $x_{\sigma(i)} x_i^{-1}$ and if $\sigma(i) = i-1$, we get y_{i-1} . Now if $i > 1$, there is always a $\sigma \in A_0$ with $\sigma(i) = i-1$ and $\sigma(i+1) = i+1$, so y_1, y_2, y_3 are all commutators, and taking $i = 4$, $\sigma(4) = 5$ and $\sigma(5) = 5$, we see that $x_5 x_4^{-1} = y_4^{-1}$ is a commutator and hence so is y_4 and so $[G_0, G_0] \supset H$ and as $[A_0, A_0] = A_0$, $[G_0, G_0] = G_0$. Moreover, as $y \rightarrow \sigma y \sigma^{-1} y^{-1}$ is linear from H_0 to H_0 it follows that any element in H_0 is a product of at most 4 commutators and as A_0 is finite it follows that there is an integer q so that any element of G_0 is a product of q commutators.

Now let T be the circle group viewed as a trivial H -module and let $B_0 = I_H^{G_0}(T)$ be the induced G_0 -module so that as a group, B_0 is a 60-dimensional torus which has a circle S of G_0 invariant elements. Let $B = B_0/S$, and we note that the topological universal covering group V of B is a 59-dimensional vector space on which G_0 operates via A_0 and every nontrivial representation of A_0 occurs with multiplicity equal to its degree. It follows immediately that in the semidirect product $G = B \cdot G_0$, $[B, A_0] = B$ and, in fact, there is an integer m such that every element of B is a product of m commutators $[b, \sigma]$, $b \in B$, $\sigma \in A_0$. Then it follows that if $K = B \cdot A_0$, $[K, K] = K$ and there is p such that every element of K is a product of p commutators. Finally note that $G = [G, G]$ and there is some integer n such that any element of G is a product of n commutators in G , and any element of K is a product of n commutators from K . Finally we note that K is compact and open in G .

We know that K has a universal covering group (in our sense) and, in fact, $\pi_1(K)$ is finite as K is a compact Lie group. We claim that G also has a fundamental group, and to do this we shall simply show that $H^2(G, T)$ is countable

and use Theorem 12. We note that $G = B \cdot G_0$ is a semidirect product and that $H^2(B, T) = (0)$ since B is a torus, and that $H^1(B, T)^{G_0} = (0)$ by construction of B , and realization of $H^1(B, T)$ as a lattice in the dual of the vector space V above. The spectral sequence for the group extension then gives, since the extension is semidirect,

$$H^2(B, T) \simeq H^2(G_0, T) \oplus H^1(G_0, H^1(B, T))$$

where $H^1(B, T) = \hat{B}$.

PROPOSITION 8. *The group $H^2(G_0, T)$ is finite.*

PROOF. We know that G_0 is an extension of A_0 by H and using the spectral sequence for this group extension it suffices to prove that each of the three groups $H^2(A_0, T)$, $H^1(A_0, H^1(H, T))$ and $H^2(H, T)^{A_0}$ is finite. The first is, in fact, of order two, the second consists of classes of crossed homomorphisms of a finite group into a 4-dimensional torus and is, consequently, finite. Finally $H^2(H, T)$ consists of skew symmetric bilinear maps $H \times H$ into T , which as a group is a torus (of dimension 6 in this case). If the fixed point set of A_0 on this group were not finite, we could construct by lifting an R valued A_0 invariant skew symmetric bilinear functional on H , and this can be extended to $H \otimes R$. But $H \otimes R$ as A_0 -module is an irreducible four-dimensional representation of A_0 and as such admits no skew symmetric bilinear invariants. We conclude then that $H^2(H, T)^{A_0}$ is finite and the proposition is proved.

PROPOSITION 9. *The group $H^1(G_0, \hat{B})$ is a free group of rank four.*

PROOF. We consider the definition of B as $B_0 = B \oplus T$ and dualize to find $\hat{B}_0 = \hat{B} \oplus Z$, the decompositions being direct as groups and as G_0 -modules. Now since B_0 is the module induced by a torus going from H to G_0 , it is immediate that $\hat{B}_0$ is the result of inducing $\hat{T} = Z$ as H -module up to G_0 . Then by Shapiro's lemma (cf. [14, Theorem 6]), $H^1(G_0, B_0) \simeq H^1(H, Z) \simeq Z^4$. Moreover, as $[G_0, G_0] = G_0$, $H^1(G_0, Z) = (0)$ and so $H^1(G_0, \hat{B}_0) = H^1(G_0, \hat{B}) \oplus H^1(G_0, Z)$ implies that $H^1(G_0, \hat{B}) \simeq Z^4$ as desired.

These two lemmas taken together say that $H^2(G, T)$ is a finitely generated group of rank four and, in particular, is countable. Thus G has a universal covering group and $\pi_1(G)$ is the sum of a finite group and a four-dimensional torus, and in any case is not discrete. We have already observed that $\pi_1(K)$ is finite and so the map $\pi_1(K) \rightarrow \pi_1(G)$ cannot have open image. This gives the desired counterexample.

Now as to the existence of universal covering groups, let G_2 be the infinite restricted direct product of countably many copies of the group G relative to the compact open subgroup K . Recall that G_2 consists of all sequences (g_n) in the

complete product with $g_n \in K$ for all but a finite number of indices n . Then G_2 is locally compact and our earlier remarks that there is an integer n such that every element of G (resp. of K) is a product of n commutators of G (resp. of K) imply that $G_2 = [G_2, G_2]$ and in fact that any element of G_2 is a product of n commutators.

Now let M be the kernel of the restriction map in cohomology: $H^2(G, T) \rightarrow H^2(K, T)$. Then M is of finite index and is therefore a finitely generated group of rank four. Then let N_1 be the infinite restricted product of copies of the group $H^2(G, T)$ relative to its subgroup M . Then N_1 has a subgroup M_1 isomorphic to the complete Cartesian product of countably many copies of M , and M_1 is of countable index in N_1 (or equal to N_1). We topologize N_1 by making M_1 open and giving it the product topology, and one may see immediately that N_1 is polonais but *not* locally compact.

Now if $a \in H^2(G_2, T)$, let a_n be the restriction of a to the n th component of the restricted product. We may argue just as in [13, Theorem 12.1] that $a \rightarrow (a_n)$ yields an algebraic isomorphism of $H^2(G_2, T)$ onto N_1 . But now since each restriction map $a \rightarrow a_n$ is continuous relative to the topologies J_m [14, Proposition 27], it is an easy map to see that the above defined map of $\underline{H}^2(G_2, T)$ into N_1 is continuous. Then an application of the closed graph theorem (both groups are polonais) shows that it is an isomorphism of topological groups, and we conclude that $\underline{H}^2(G_2, T)$ is not locally compact, and hence that G_2 cannot have a universal covering group. This is our desired example.

REFERENCES

1. L. Auslander and C. C. Moore, *Unitary representations of solvable Lie groups*, Mem. Amer. Math. Soc. No. 62 (1966). MR 34 #7723.
2. F. Browder, *On the iteration of transformations in noncompact minimal dynamical systems*, Proc. Amer. Math. Soc. 9 (1958), 773–780. MR 20 #3456.
3. A. Guichardet, *Sur la cohomologie des groupes topologiques*. I, II, Bull. Sci. Math. (2) 95 (1971), 161–176; *ibid.* (2) 96 (1972), 305–322. MR 46 #6385; 49 #5219.
4. B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. No. 127 (1972).
5. G. P. Hochschild and J. P. Serre, *Cohomology of group extensions*, Trans. Amer. Math. Soc. 74 (1973), 110–134.
6. K. Jacobs, *Neuere Methoden und Ergebnisse der Ergodentheorie*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Heft 29, Springer-Verlag, Berlin, 1960. MR 23 #A1000.
7. R. R. Kallman, *Groups of inner automorphisms of von Neumann algebras*, J. Functional Analysis 7 (1971), 43–60. MR 43 #5317.
8. G. W. Mackey, *Induced representations of locally compact groups*. I, Ann. of Math. (2) 55 (1952), 101–139. MR 13, 434.
9. L. Michel, *Sur les extensions centrales du groupe de Lorentz inhomogène connexe*, Nuclear Phys. 57 (1965), 356–385. MR 35 #1699.
10. C. C. Moore, *Extensions and low dimensional cohomology theory of locally compact groups*. I, Trans. Amer. Math. Soc. 113 (1964), 40–63. MR 30 #2106.
11. ———, *Extensions and low dimensional cohomology theory of locally compact*

groups. II, Trans. Amer. Math. Soc. 113 (1964), 64–86. MR 30 #2106.

12. ———, *Restrictions of unitary representations to subgroups and ergodic theory: Group extensions and group cohomology*, Group Representations in Math. and Phys. (Battelle Seattle, 1969 Rencontres), Lecture Notes in Phys., vol. 6, Springer, Berlin, 1970, pp. 1–35. MR 43 #4955.

13. ———, *Group extensions of p -adic and adelic linear groups*, Inst. Hautes Études Sci. Publ. Math. No. 35 (1968), 157–222. MR 39 #5575.

14. ———, *Group extensions and cohomology for locally compact groups*. III, Trans. Amer. Math. Soc. 221 (1976), 1–33.

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